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General Properties of Realistic Neural Network Dynamics

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Abstract—The role of symmetry in the dynamics of the Realistic Neural Network model is studied. The phase diagram for Symmetric Realistic Neural Network with an external input of the constrained in time action is constructed. The oscillation regimes are investigated. For RNN of general form, the mean-field approximation is obtained.

Keywords—Neural networks, Mean-field approximation.

1. INTRODUCTION

Models of neural networks fall into two main classes: Ising-like models and Adaptive Filters, for which a lot of work has been done. In comparison with these models, the Realistic Neural Network model suggested by Kropotov [1–3] seems to be more relevant to the physiology of the brain due to the consideration of electrochemical processes of synaptic depression and initiation. One may hope that in the frame of this model, it is possible to study the essential phenomena such as burst-like behavior, binding mechanism, short- and long-term memory, temporal coding of information, and so on [4,5]. Apart from it, RNN is also interesting as an artificial neural network, a development of which would be useful for study of industrial devices for treatment and storage of information. In the present paper, we establish the general features of the RNN dynamics.

2. THE RNN MODEL

The RNN equations are of the form

$$P_i(k+1) - P_i(k) = -\alpha P_i(k) + \sum_j W_{ij}(k) N_j(k) - \beta N_i(k) + S_i(k) \tag{1}$$

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$$W_{ij}(k) = \left(x_i^1(k) + x_i^2(k)\right) W_{ij}^0 \tag{2}$$

$$x_i^1(k+1) - x_i^1(k) = -A_1 x_i^1(k) + B_1 N_i(k) + C_1 \tag{3}$$

$$x_i^2(k+1) - x_2^1(k) = -A_2 x_i^2(k) - B_2 N_i(k) + C_2 \tag{4}$$

$$N_i(k) = \theta\left(P_i(k) - h_i\right). \tag{5}$$

Here $P_i(k), N_i(k)$ are the membrane potential and the activity of the i^{th} neuron at the time moment k. The function $\theta(x)$ is the threshold one: $\theta(x) = 1$ if $x > 0$, $\theta(x) = 0$ if $x < 0$. The synaptic processes of potentiation and depression are modelled by the dynamics of the variables $x_i^1(k)$,$x_i^2(k)$. In accordance with experimental data, the synaptic strength is defined by two different processes: short-term (dynamic-variables $x_i^1(k)$, $x_i^2(k)$) and long-term ones (static-matrix W_{ij}^0). The parameters $\alpha, \beta, h, A, B, C$ are supposed to be constant and positive. The membrane potential P is a rapid variable in comparison with the variables x: $A \simeq B$, $\alpha \simeq \beta$, $\alpha \simeq 10A$. The function $S(k)$ is assumed to be given and describes in the RNN an external input.

3. EQUATIONS FOR NEURAL ACTIVITIES

For convenience of investigations, we shall reduce the RNN equations to a system of nonlinear equations for neuron activities only. Let us consider the linear equation of the form

$$A(k+1) = \alpha A(k) + B(k), \qquad k \geq 0, \quad A(0) = A_0. \tag{6}$$

It is easy to see that its solution is

$$A(k) = \sum_{l=0}^{k-1} B(l)\alpha^{k-l-1} + \alpha^k A_0. \tag{7}$$

With the help of (6),(7), we can solve equations (1),(3),(4) and obtain $P_i(k)$, $x_i^1(k)$, $x_i^2(k)$ as the functions of activities $N_i(k)$, input $S_i(k)$, interconnection matrix $W_{ij}(k)$ and constant RNN parameters. Expressing $W_{ij}(k)$ through $x_i^1(k)$ and $x_i^2(k)$ with the help of (2) and using (5), we obtain the equations

$$\begin{aligned} N_i(k+1) = \theta \Bigg[& P_i(0)(1-\alpha)^{k+1} \\ & + \sum_{l=0}^{k} (1-\alpha)^{k-l} \left(\sum_j W_{ij}(l) N_j(l) - \beta N_i(l) + S_i(l) \right) - h_i \Bigg], \end{aligned} \tag{8}$$

where

$$\begin{aligned} W_{ij}(l) = W_{ij}^0 \Bigg(& (1-A_1)^l X_{1j}(0) + (1-A_2)^l X_{2j}(0) \\ & + \sum_{m=0}^{l-1} \left[(1-A_1)^{l-m-1} B_1 - (1-A_2)^{l-m-1} B_2 \right] N_j(m) \\ & + \sum_{m=0}^{l-1} \left[(1-A_1)^{l-m-1} C_1 + (1-A_2)^{l-m-1} C_2 \right] \Bigg). \end{aligned} \tag{9}$$

Thus we have gotten the system of nonlinear equations for neural activities $N_i(k)$ only. It will be the basis of our later considerations.

4. THE ROLE OF SYMMETRY OF STATIC INTERCONNECTIONS SYMMETRY IN THE RNN

One of the important problems in the neurophysiology is the coherent behavior of neuron ensembles [5]. At present, the mechanism of this phenomenon is not clear. We shall show that in the RNN model such effect may appear as the result of the symmetry of static interconnections matrix W_{ij}^0. Namely, if the RNN with N neurons has a subset of M neurons such as the matrix W_{ij}^0, initial conditions $P_i(0)$ and inputs S_i are invariant under a renumeration of those neurons, then the system of equations for the RNN can be reduced to the system of $N - M - 1$ equations.

Let the RNN be a system of N neurons such that it can be divided into K subsets of neurons $\{1\}, \{2\}, \dots, \{k\}$ with the following property—the RNN equations and initial conditions are invariant under any renumeration of neurons which conserve the subset $\{1\}, \{2\}, \dots, \{k\}$. Then dynamical behavior of the all neurons which belong to the subset $\{l\}$, $1 \le l \le k$ is the same. It means that we can replace N equations for considered RNN with the system of ones for RNN with K neurons and the interconnection matrix of the following form:

$$W_{ab} = \sum_{j \in \{b\}} W_{ij}, \qquad i \in \{a\}. \tag{10}$$

In (10), the summation is meant to be performed over all the numbers j which belong to the subset $\{b\}$. Thus we have a new RNN in which every neuron with number a represents all neurons of the subset $\{a\}$ in the original RNN. From (10), it follows that

$$W_{ab} = W_{ij} N_b, \qquad i \in \{a\}, \quad j \in \{b\}, \tag{11}$$

where N_b is a number of neurons in a subset b. So the symmetry in the RNN enlarges effectively of the interconnection strengths.

5. SYMMETRIC RNN

If the number K of subsets mentioned above is 1, such RNN will be called the fully symmetrical RNN (SRNN). The dynamical equations for the SRNN are of the form

$$P(k+1) - P(k) = -\alpha P(k) + W(k)N(k) - \beta N(k) + S(k) \tag{12}$$

$$W(k) = (x_1(k) + x_2(k))\, W^0 \tag{13}$$

$$x_1(k+1) - x_1(k) = -A_1 x_1(k) + B_1 N(k) + C_1 \tag{14}$$

$$x_2(k+1) - x_2(k) = -A_2 x_2(k) - B_2 N(k) + C_2 \tag{15}$$

$$N(k) = \theta(P(k) - h), \tag{16}$$

where

$$W^0 = \frac{\sum_{ij} W_{ij}^0}{N}. \tag{17}$$

In (17), N denotes the number of neurons in the SRNN.

Instead of (8), we have the following equation for neuronal activity in SRNN:

$$N(k+1) = \theta\left[P(0)(1-\alpha)^{k+1} + \sum_{l=0}^{k} (1-\alpha)^{k-l} \left(W(l)N(l) - \beta N(l) + S(l)\right) - h\right]. \tag{18}$$

Here

$$\begin{aligned} W(l) = W^0 \Bigg(& (1-A_1)^l x_1(0) + (1-A_2)^l x_2(0) \\ & + \sum_{m=0}^{l-1} \left[(1-A_1)^{l-m-1} B_1 - (1-A_2)^{l-m-1} B_2\right] N(m) \\ & + \sum_{m=0}^{l-1} \left[(1-A_1)^{l-m-1} C_1 + (1-A_2)^{l-m-1} C_2\right] \Bigg). \end{aligned} \tag{19}$$

It is easy to prove the following lemma.

LEMMA. *If* $\alpha < 1$, $A_1 < 1$, $A_2 < 1$, $S(k) = 0$ *for* $k > t$ *and* $N(k_0) = 0$ *for* $k_0 > t$, *then* $N(k) = 0$ *for all* $k \geq k_0$.

The proof is obtained directly from equation (12): for $k = k_0$, it has the form

$$P(k+1) - P(k) = -\alpha P(k).$$

Hence, $N(k) = 0$ for $k = k_0 + 1$.

In virtue of the lemma, only two types of dynamical behavior can exist in SRNN if $S(k) = 0$ for $k > t$ (external input of the constrained in time action): the asymptotic of $N(k)$ is $N = 0$ or $N = 1$.

If $N(k) = 1$ for $k \geq 0$, then from equations (18),(19), we get

$$W(l) = W^0 \left((1 - A_1)^l x_1(0) + (1 - A_2)^l x_2(0) + \frac{B_1 + C_1}{A_1} \left[1 - (1 - A_1)^l \right] - \frac{B_2 - C_2}{A_2} \left[1 - (1 - A_2)^l \right] \right) \tag{20}$$

and

$$\begin{aligned} \sum_{l=0}^{k} (1-\alpha)^{k-l} \left(W(l)N(l) - \beta N(l) + S(l) \right) - h \\ = \left[\left(\frac{B_1 + C_1}{A_1} - \frac{B_2 - C_2}{A_2} \right) - \beta \right] W_0 \frac{1 - (1-\alpha)^{k+1}}{\alpha} \\ + \left(\frac{B_1 + C_1}{A_1} + x_1(0) \right) \frac{(1 - A_1)^{k+1} (1-\alpha)^{k+1}}{A_1 - \alpha} \\ + \left(\frac{C_2 - B_2}{A_2} + x_2(0) \right) \frac{(1 - A_2)^{k+1} (1-\alpha)^{k+1}}{A_2 - \alpha} + (1-\alpha)^k \sum_{l=0}^{k_0} S(l) (1-\alpha)^{-l} - h. \end{aligned}$$

Then from (16), for $k \to \infty$, we obtain

$$1 = \theta \left[\left(\frac{B_1 + C_1}{A_1} - \frac{B_2 - C_2}{A_2} - \beta \right) \frac{W_0}{\alpha} - h \right]. \tag{21}$$

Hence, $N(k \to \infty) = 0$ for

$$\left(\frac{C_1 + B_1}{A_1} + \frac{C_2 - B_2}{A_2} \right) W_0 - \beta < \alpha h \tag{22}$$

and $N(k \to \infty) = 1$ for

$$\left(\frac{C_1 + B_1}{A_1} + \frac{C_2 - B_2}{A_2} \right) W_0 - \beta > \alpha h, \tag{23}$$

correspondingly.

These inequalities define the phase diagram of SRNN under conditions: $\alpha < 1$, $A_1 < 1$, $A_2 < 1$, $S(k) = 0$ for $k > t$. Thus, we have proved that in the SRNN, there are only two types of dynamical behavior with trivial fixed point asymptotes. For arbitrary initial conditions, the statement has been verified by the numerical experiments (see Figure 1, where the control parameter ϕ is the left side of the inequalities after replacing the term αh to the left). The network consisted of by 10 neurons, and the parameters were the following: $W_{ij} = 0.01(1 - \delta_{ij})$, $W_0 = 0.1$, $\alpha = 0.8$, $\beta = 0.7$, $A_1 = B_1 = C_1 = 0.01$, $A_2 = B_2 = C_2 = 0.02$, $h = 0.0$. This set of parameters was a basic one for our numerical calculations. Below we shall mention only the parameters which have been changed. In this case, we have been changing ϕ by modifying β.

As the initial potential P_0 changes, such behavior remains the same. For example, for SRNN in the excitation phase changes of P_0 on several orders did not impact on the average activity.

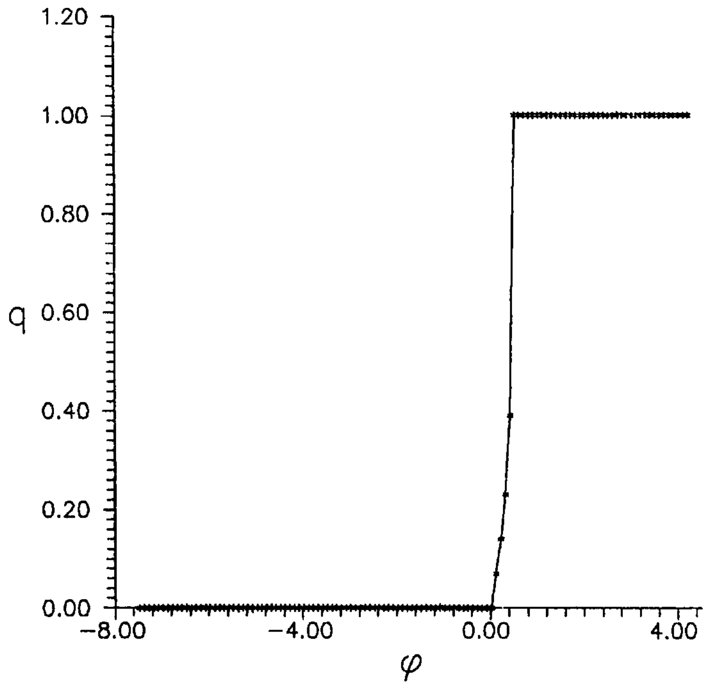


Figure 1. Dependence of an average activity in the SRNN on the control parameter.

6. OSCILLATIONS IN SRNN

Let us consider the case of periodical $S(k)$: $S(k+t) = S(k)$ for $k > 0$. For convenience of calculations, let us introduce the generating functions

$$s(z) = \sum_{l=0}^{\infty} S(l) z^l, \qquad n(z) = \sum_{l=0}^{\infty} N(l) z^l,$$

where z is the complex variable. The series for $n(z), s(z)$ are obviously convergent for $|z| < 1$. For other z, the functions are thought of as the analytical extensions. The expressions $S(l), N(l)$, through generating functions, are of the form

$$S(l) = \oint \frac{dz}{2\pi i} \frac{s(z)}{z^{l+1}}, \qquad N(l) = \oint \frac{dz}{2\pi i} \frac{n(z)}{z^{l+1}}. \tag{24}$$

The integration contours in (24) are the circles on the complex plane of a radius $R < 1$ with centers in the point $z = 0$. In virtue of periodicity of $S(l)$,

$$s(z) = \frac{\sigma(z)}{1 - z^t},$$

where

$$\sigma(z) = \sum_{l=0}^{t-1} S(l) z^l.$$

We assume that the solution of equation (18) has the form

$$N(k) = N_1(k) + N_2(k),$$

where

$$N_1(k+t) = N_1(k), \qquad N_2(k) = 0,$$

for

$$k \geq k_0.$$

Then,

$$n(z) = n_1(z) + n_2(z),$$

where

$$n_1(z) = \frac{\nu(z)}{1 - z^t}, \qquad \nu \equiv \sum_{l=0}^{t-1} N_1(l) z^l, \qquad n_2(z) = \sum_{l=0}^{k_0} N_2(l) z^l.$$

In terms of the functions $s(z)$ and $n(z)$ in expressions in the right-hand side of the equation (18) for the activity, can be represented in the following form:

$$G_k(\alpha, A) \equiv \sum_{l=0}^{k} (1-\alpha)^{k-l} N(l) \sum_{m=0}^{l-1} (1-A)^{l-m-1} N(m)$$
$$= \oint \oint \frac{dz\, dz'\, n(z) n(z')}{(2\pi i)^2 z^k z'^{k+1} ((1-A)z - 1)(1-\alpha) zz' - 1},$$

$$F_k(\alpha) \equiv \sum_{l=0}^{k} (1-\alpha)^{k-l} N(l) = \oint \frac{dz\, n(z)}{2\pi i z^{k+1} (1 - \alpha - z)}$$

and

$$R_k(\alpha) \equiv \sum_{l=0}^{k} (1-\alpha)^{k-l} S(l) = \oint \frac{dz\, s(z)}{2\pi i z^{k+1} (1 - \alpha - z)}.$$

If $0 \leq l < t$, $0 < \alpha < 1$, $0 < A < 1$, then

$$\begin{aligned} g(l, \alpha, A) &\equiv \lim_{m\to\infty} G(mt + l, \alpha, A) \\ &= (1-\alpha)^l \left[\frac{\nu(1/(1-A))\,(1-A)^{t-1}}{1-(1-A)^t} \left[\frac{\nu((1-A)/(1-\alpha))\,(1-\alpha)^t}{1-(1-\alpha)^t} \right.\right. \\ &\quad \left.\left. + \nu_l \left(\frac{1-A}{1-\alpha} \right) \right] + \oint \frac{dz\, \nu(z)}{2\pi i\,(1-(1-A)z)} \left[\frac{\nu(1/(1-\alpha)z)\,(1-\alpha)^t}{1-(1-\alpha)^t} + \nu_l \left(\frac{1}{(1-\alpha)z} \right) \right] \right] \end{aligned} \tag{25}$$

$$f(l, \alpha) \equiv \lim_{m\to\infty} F(mt + l, \alpha) = (1-\alpha)^l \left[\frac{(1-\alpha)^{t-1} \nu(1/(1-\alpha))}{1-(1-\alpha)^t} + \nu_l \left(\frac{1}{1-\alpha} \right) \right] \tag{26}$$

$$r(l, \alpha) \equiv \lim_{m\to\infty} R(mt + l, \alpha) = (1-\alpha)^l \left[\frac{\sigma(1/(1-\alpha))(1-\alpha)^{t-1}}{1-(1-\alpha)^t} + \sigma_l \left(\frac{1}{1-\alpha} \right) \right]. \tag{27}$$

Here

$$\nu_l(z) = \sum_{i=0}^{l} N_l(i) z^i, \qquad \sigma_l(z) = \sum_{i=0}^{l} S_l(i) z^i.$$

In virtue of (25),(26),(27), one obtains from (18) the equation for $N_1(k)$ of the following form:

$$\begin{aligned} N_1(k+1) = \theta \Bigg(& W_0 \left(B_1 g(k, \alpha, A_1) - B_2 g(k, \alpha, A_2) \right) \\ & + \left(\frac{C_1}{A_1} + \frac{C_2}{A_2} - \beta \right) f(k, \alpha) + r(k, \alpha) - h \Bigg), \end{aligned} \tag{28}$$

for

$$0 \leq k < t, \quad N_1(t) = N_1(0).$$

This equation describes the oscillations in SRNN influenced by the periodical input $S(k)$. If S is constant, $r(k, \alpha) = S\alpha^{-1}$. In this case, the equation (28) for $N(k)$ has the form

$$N_1(k+1) = \theta \left(W_0 \left(B_1 g(k, \alpha, A_1) - B_2 g(k, \alpha, A_2) \right) + \left(\frac{C_1}{A_1} + \frac{C_2}{A_2} - \beta \right) f(k, \alpha) - h_{\text{eff}} \right), \tag{29}$$

where

$$h_{\text{eff}} \equiv h - S\alpha^{-1}.$$

7. CONSTANT INPUT

The oscillations for the constant input can be considered as an important characteristic of the SRNN dynamics. The simplest are the oscillations arising when $B_1 g(k,\alpha,A_1) = B_2 g(k,\alpha,A_2)$ and the corresponding equations (29) are as follows:

$$N_1(k+1) = \theta\left(Lf(k,\alpha) - h_{\text{eff}}\right). \tag{30}$$

Here $L \equiv (C_1/A_1) + (C_2/A_2) - \beta$. The function $f(k,\alpha)$ is positive for $0 \le k < t$. As it has been shown, the oscillations can arise for $h_{\text{eff}} < 0$ only. Hence, L must be in this case negative, and (30) can be written in the form

$$N_1(k+1) = \theta\left(M - f(k,\alpha)\right), \tag{31}$$

where $M \equiv -h_{\text{eff}} L^{-1}$.

Let us find the solution of (31) in the form

$$N(0) = N(2) = \cdots = N(d-1) = 0, \qquad N(d) = N(d+1) = \cdots = N(t-1) = 1.$$

For such $N(k)$,

$$f(k,\alpha) = (1-\alpha)^{k+1}\frac{1-(1-\alpha)^{t-d}}{\alpha-(1-\alpha)^t} - \theta(k-d)\frac{1-(1-\alpha)^{k+1-d}}{\alpha}. \tag{32}$$

Then it follows from (32) that for $d > 1$,

$$f(d-2,\alpha) \le M < f(t-2,\alpha).$$

This inequality has the solution for $d = t-1$ only, and the constant M must satisfy the following constrains:

$$\frac{(1-\alpha)^{t-2}}{1-(1-\alpha)^t} > M \ge \frac{(1-\alpha)^{t-1}}{1-(1-\alpha)^t}.$$

For the solution of (31) with $d = 1$, the corresponding constrains for M are as follows:

$$\frac{1}{\alpha} - \frac{(1-\alpha)^{t-1}}{1-(1-\alpha)^t} > M \ge \frac{1}{\alpha} - \frac{(1-\alpha)^{t-2}}{1-(1-\alpha)^t}.$$

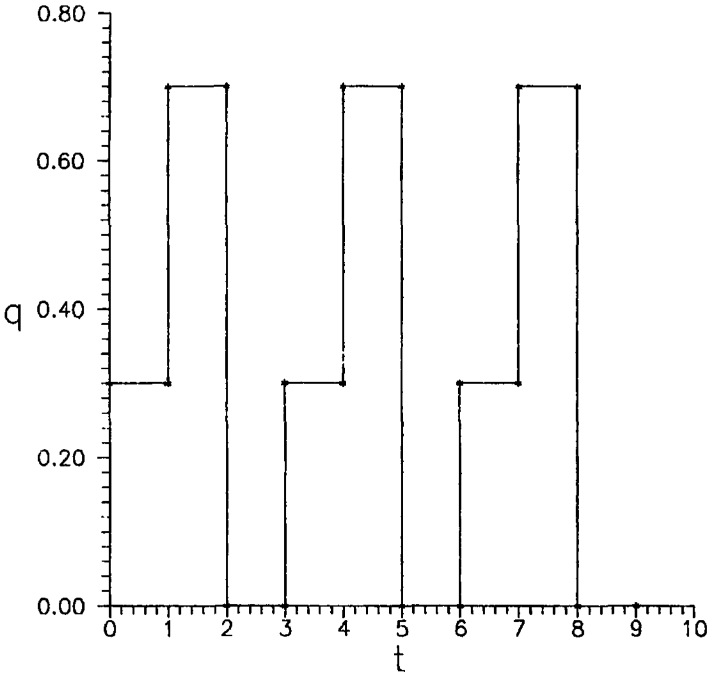
For $0 > H$, $N = 0$, and for $H > \alpha^{-1}$, $N = 1$; hence, there are no other oscillation solutions of equation (30).

8. DISCUSSION

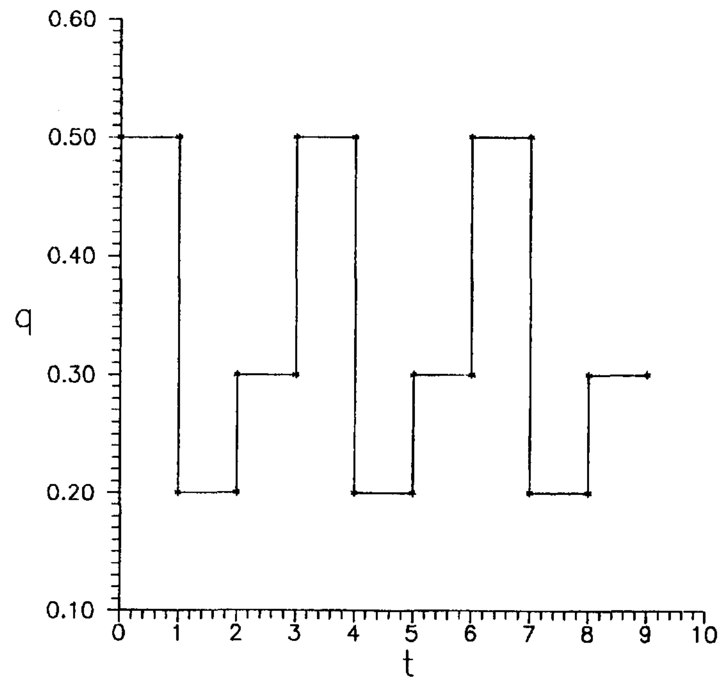
We have described the types of the SRNN dynamics in the presence of an external input of the constrained in time action. For more complex RNN with $W_{ij}^0 \ne W_{ij}^{\text{sym}}$, one may hope that phase diagrams (22),(23) constructed by us are an approximation, if the parameter W_0 is calculated as

$$W_0 = \frac{\sum_{ij} W_{ij}^0}{n}.$$

It can be considered as the mean field approximation for RNN and would be useful in numerical experiments by investigations of the RNN. Such experiments which have been done by us verified the statement following from the analytical studies, that in the SRNN, there are only two possible types of oscillations of neural activities. Namely, let us consider the oscillations with the period k and let q_{eff} be a sum of neural activity on the period. It has been proven that there can be only two possibilities: $q_{\text{eff}} = 1$ or $q_{\text{eff}} = k-1$. Numerical experiments have shown that this property



(a)

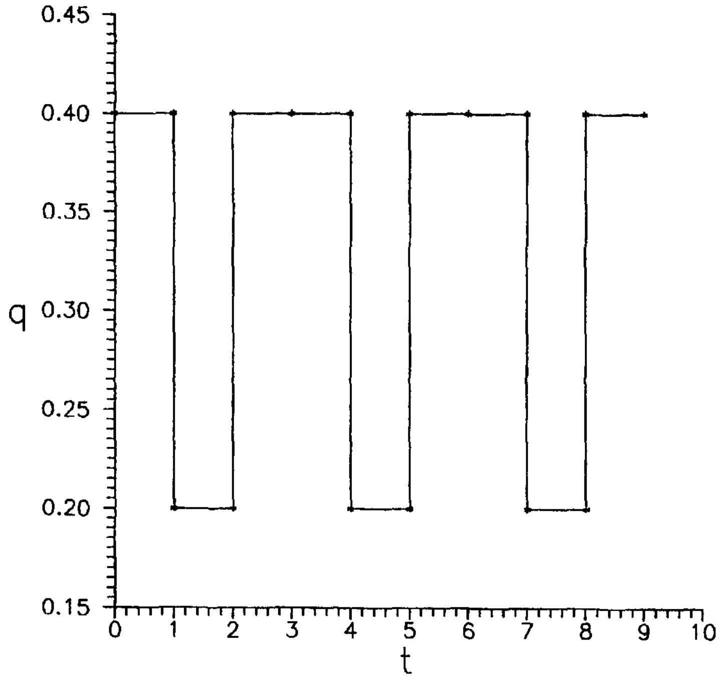


(b)

Figure 2. Different types of oscillations of average neuronal activity with period 3 in SRNN with broken symmetry.

of the oscillation dynamics remains the same for the nonsymmetrical RNN (see Figure 2, where $k = 3$).

It is interesting to note that in the case of constant external signal, the frequency of oscillations in SRNN depends on the value of the signal in such a way that more stable (in respect to small variations of the signal) are the oscillations with the frequency 1/2, 1/3, 2/3, and so on (see Figure 3). In other words, the stability of the oscillations in the SRNN decreases with the



(c)

Figure 2 (cont.)

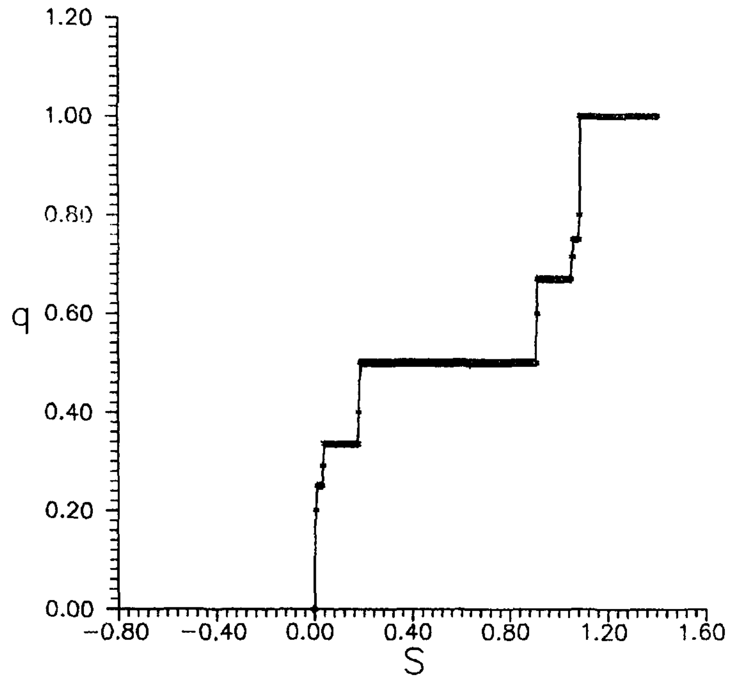


Figure 3. Dependence of the average activity on the value of external signal.

decreasing of the frequency so that only a few oscillations regimes could be practically achieved in case of "real" (nonstable) input. It is interesting to note that such a feature is inherent to real brain rhythms.

In conclusion, we would like to stress that our analytic results are explicit for SRNN and would be considered as the mean-field approximation for the general case. We suppose that the main results obtained for the SRNN would be useful for studies of models with not very strong broken symmetry of the static interconnection matrix (neuron structures like surfaces, lines, etc., with approximate translation invariance). By our opinion, such quasisymmetry of interconnection matrices would be one of the reasons for the appearance of burst-like behavior in the dynamics of neural networks.

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